Methods in Calculus Cheat Sheet

This cheat sheet explores three useful ideas in calculus: evaluating n^{th} derivatives using Leibnitz's theorem, evaluating certain indeterminate limits using L'Hospital's rule, and finding definite and indefinite integrals using the Weierstrass substitution.

Leibnitz's theorem and n^{th} derivatives

Leibnitz's theorem expands upon the use of the product rule for derivatives. Given $y = uv$, where u and v are functions of a variable x,

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

By reapplying the product rule to $\frac{dy}{dx}$, we can find higher derivatives of this function. We do so by using the product rule on $u\frac{dv}{dx}$ and $v\frac{du}{dx}$,

$$\frac{d^2y}{dx^2} = \left(\frac{du}{dx}\frac{dv}{dx} + u\frac{d^2v}{dx^2}\right) + \left(\frac{dv}{dx}\frac{du}{dx} + v\frac{d^2u}{dx^2}\right)$$

$$= u\frac{d^2v}{dx^2} + 2\frac{du}{dx}\frac{dv}{dx} + v\frac{d^2u}{dx^2}$$

The process of applying the product rule to all the terms can be repeated to obtain results for even higher derivatives,

$$\frac{d^3y}{dx^3} = u\frac{d^3v}{dx^3} + 3\frac{du}{dx}\frac{d^2v}{dx^2} + 3\frac{d^2u}{dx^2}\frac{dv}{dx} + v\frac{d^3u}{dx^3}$$

$$\frac{d^4y}{dx^4} = u\frac{d^4v}{dx^4} + 4\frac{du}{dx}\frac{d^3v}{dx^3} + 6\frac{d^2u}{dx^2}\frac{d^2v}{dx^2} + 4\frac{d^3u}{dx^3}\frac{dv}{dx} + v\frac{d^4u}{dx^4}$$

It can be observed that the coefficients of the terms follow the binomial expansion and can thus be represented by $\binom{n}{k} = \frac{n!}{k!(n-k)!}$.

Leibnitz's Theorem uses this observation to provide a general formula for the n^{th} derivative of the product of 2 functions. Now, given that $y = uv$ (where u and v are functions of a variable x), Leibnitz's Theorem states,

$$\frac{d^ny}{dx^n} = \sum_{k=0}^{n}\binom{n}{k}\frac{d^ku}{dx^k}\frac{d^{n-k}v}{dx^{n-k}}$$

Example 1: Use Leibnitz's theorem to calculate $\frac{d^4y}{dx^4}$ for $y = e^{2x}\cosh x$.

Write down u and v.	$y = uv$, where $u = e^{2x}$ and $v = \cosh x$
Calculate each derivative for u and v.	$u = e^{2x} \Rightarrow \frac{du}{dx} = 2e^{2x} \Rightarrow \frac{d^2u}{dx^2} = 4e^{2x}$ $\Rightarrow \frac{d^3u}{dx^3} = 8e^{2x} \Rightarrow \frac{d^4u}{dx^4} = 16e^{2x}$ $v = \cosh x \Rightarrow \frac{dv}{dx} = \sinh x \Rightarrow \frac{d^2v}{dx^2} = \cosh x$ $\Rightarrow \frac{d^3v}{dx^3} = \sinh x \Rightarrow \frac{d^4v}{dx^4} = \cosh x$
Use Leibnitz's theorem to write down the general form for the fourth derivative of $y = e^{2x}\cosh x$.	$\frac{d^4y}{dx^4} = u\frac{d^4v}{dx^4} + 4\frac{du}{dx}\frac{d^3v}{dx^3} + 6\frac{d^2u}{dx^2}\frac{d^2v}{dx^2}$ $+ 4\frac{d^3u}{dx^3}\frac{dv}{dx} + \frac{d^4u}{dx^4}v$
Substitute the previously calculated derivatives into this general form and simplify.	$\frac{d^4y}{dx^4} = e^{2x}\cosh x + 8e^{2x}\sinh x$ $+ 24e^{2x}\cosh x$ $+ 32e^{2x}\sinh x$ $+ 16e^{2x}\cosh x$ $\Rightarrow \frac{d^4y}{dx^4} = 41e^{2x}\cosh x + 40e^{2x}\sinh x$

When asked to apply Leibnitz's theorem to equations of the form $y = \frac{a}{b}$, we can make the equation the product of 2 functions $u = a$ and $v = \frac{1}{b}$.

L'Hospital's rule

As encountered when looking at Taylor series, some limits are of indeterminate form. We can apply L'Hospital's rule to tackle limits of the indeterminate forms $\frac{0}{0}$ or $\frac{\pm\infty}{\pm\infty}$. These forms arise when trying to find the limit of a function of the form $\frac{f(x)}{g(x)}$ (where $f(x)$ and $g(x)$ are differentiable functions) at a location where both $f(x)$ and $g(x)$ tend to 0 or $\pm\infty$.

L'Hospital's rule states, given that,

$$\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0 \ \text{ or } \ \lim_{x\to a} f(x) = \pm\infty \text{ and } \lim_{x\to a} g(x) = \pm\infty$$

And that $\lim_{x\to a}\frac{f(x)}{g(x)}$ exists,

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$$

Example 2: Evaluate the limit $\lim_{x\to 2}\frac{x^2+x-6}{x^2-4}$.

First, check that L'Hospital's rule can be applied by calculating the limit by substitution.	$\lim_{x\to 2}\frac{x^2+x-6}{x^2-4} = \frac{4+2-6}{4-4}$ $= \frac{0}{0}$ (Indeterminate form) Thus, L'Hospital's rule can be applied.
Write down $f(x)$ and $g(x)$ and find their derivatives.	Let $f(x) = x^2 + x - 6 \Rightarrow f'(x) = 2x + 1$ Let $g(x) = x^2 + 4 \Rightarrow g'(x) = 2x$
Apply L'Hospital's rule.	Applying L'Hospital's rule $\lim_{x\to 2}\frac{x^2+x-6}{x^2-4} = \lim_{x\to 2}\frac{2x+1}{2x} = \frac{5}{4}$

It is also possible for the limits of the derivatives to be indeterminate. In this case, L'Hospital's rule needs to be applied again.

Example 3: Evaluate the limit $\lim_{x\to\infty}\frac{e^x}{x^2}$.

First, check that L'Hospital's rule can be applied by calculating the limit by substitution.	$\lim_{x\to\infty}\frac{e^x}{x^2} = \frac{e^\infty}{\infty^2}$ $= \frac{\infty}{\infty}$ (Indeterminate form) Thus, L'Hospital's rule can be applied.
Write down $f(x)$ and $g(x)$ and find their derivatives.	Let $f(x) = e^x \Rightarrow f'(x) = e^x$ Let $g(x) = x^2 \Rightarrow g'(x) = 2x$
Apply L'Hospital's rule.	Applying L'Hospital's rule: $\lim_{x\to\infty}\frac{e^x}{x^2} = \lim_{x\to\infty}\frac{e^x}{2x}$ $= \frac{\infty}{\infty}$ (Indeterminate form) Hence, L'Hospital's rule needs to be applied again.
Find the second derivatives of $f(x)$ and $g(x)$.	$f''(x) = e^x$ $g''(x) = 2$
Apply L'Hospital's rule again.	$\lim_{x\to\infty}\frac{e^x}{x^2} = \lim_{x\to\infty}\frac{e^x}{2} = \infty$

The following result is also useful to note for some exam questions,

$$\text{If } \lim_{x\to a} f(x) \text{ exists, then } \lim_{x\to a} e^{f(x)} = e^{\lim_{x\to a} f(x)}$$

The Weierstrass substitution

To simplify some trigonometric integrals, we can use the Weierstrass substitution along with the t-formulae. The Weierstrass substitution is known to be $t = \tan\frac{x}{2}$, where we also replace dx with $\frac{2}{1+t^2}dt$.

After using the Weierstrass function, most of the time you are left with a rational function (a fraction with polynomials). These can be integrated using partial fractions, or another appropriate technique.

The substitution can then finally be reversed to obtain the final answer. Weierstrass functions are especially useful with evaluating an integral with a $\cos x$ or $\sin x$ in the denominator.

Example 4: Evaluate $\int \sec x \, dx$ using the Weierstrass substitution.

Use the t-substitution to transform $\sec x$ into algebraic form.	If $t = \tan\frac{x}{2}$, $\sec x = \frac{1+t^2}{1-t^2}$ $dx = \frac{2}{(1+t^2)}dt$
Rewrite and evaluate the integral by using partial fractions.	$\int \sec x \, dx = \int \frac{1+t^2}{1-t^2} \times \frac{2}{1+t^2}dt$ $= \int \frac{2}{1-t^2}dt$ $= \int \frac{1}{1+t} + \frac{1}{1-t}\,dt$ $= \ln\lvert 1+t\rvert - \ln\lvert 1-t\rvert + c$
Simplify and undo the substitution. On the second line, we have essentially multiplied by 1 to aid us in our simplification. Recall that $\sec x = \frac{1+t^2}{1-t^2}$ and $\tan x = \frac{2t}{1-t^2}$ according to the t-formulae.	$= \ln\left\lvert\frac{1+t}{1-t}\right\rvert + c$ $= \ln\left\lvert\frac{1+t}{1-t} \times \frac{1+t}{1+t}\right\rvert + c$ $= \ln\left\lvert\frac{(1+t)^2}{1-t^2}\right\rvert + c$ $= \ln\left\lvert\frac{1+t^2+2t}{1-t^2}\right\rvert + c$ $= \ln\left\lvert\frac{1+t^2}{1-t^2} + \frac{2t}{1-t^2}\right\rvert + c$ $= \ln\lvert\sec x + \tan x\rvert + c$

Example 5: Use the substitution $t = \tan\frac{x}{2}$ to evaluate the integral $\int_0^{\frac{3\pi}{4}}\frac{2}{7+8\cos x}dx$.

Use the t-substitution to transform the integral into algebraic form.	$\int_0^{\frac{3\pi}{4}}\frac{2}{7+8\cos x}dx$ $= \int_0^{\frac{3\pi}{4}}\frac{2}{7+8\left(\frac{1-t^2}{1+t^2}\right)}dx$
Change the variable of integration and the limits.	$dx = \frac{2}{1+t^2}dt$ Thus, the integral becomes, $\int_0^{\frac{3\pi}{4}}\frac{2}{7+8\left(\frac{1-t^2}{1+t^2}\right)} \times \frac{2}{1+t^2}dt$ $= \int_0^{\frac{3\pi}{4}}\frac{4}{7(1+t^2)+8(1-t^2)}dt$ $= \int_0^{\frac{3\pi}{4}}\frac{4}{7+7t^2+8-8t^2}dt$ $= \int_0^{\frac{3\pi}{4}}\frac{4}{15-t^2}dt$ $\tan\frac{0}{2} = 0$, $\quad \tan\frac{\left(\frac{3\pi}{4}\right)}{2} = \sqrt{2}+1$ $= \int_0^{1+\sqrt{2}}\frac{4}{15-t^2}dt$
Perform partial fraction decomposition and then evaluate the integral.	$\int_0^{1+\sqrt{2}}\frac{4}{15-t^2}dt$ $= \int_0^{1+\sqrt{2}}\frac{2}{\sqrt{15}(t+\sqrt{15})} + \frac{2}{\sqrt{15}(\sqrt{15}-t)}dt$ $= \left[\frac{2\ln(\lvert t+\sqrt{15}\rvert) - 2\ln(\lvert t-\sqrt{15}\rvert)}{\sqrt{15}}\right]_0^{1+\sqrt{2}}$
Substitute the limits in and evaluate.	$= 0.754$(3 sig. figs)